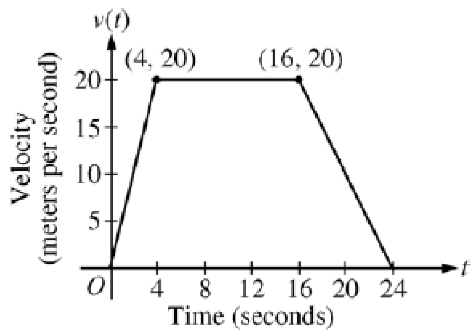
Conceptual Exercises on the Fundamental Theorem of Calculus

1.



A car is traveling on a straight road. For $0 \le t \le 24$ seconds, the car's velocity $v(t)$, in meters per second, is modeled by the piecewise-linear function defined by the graph above.

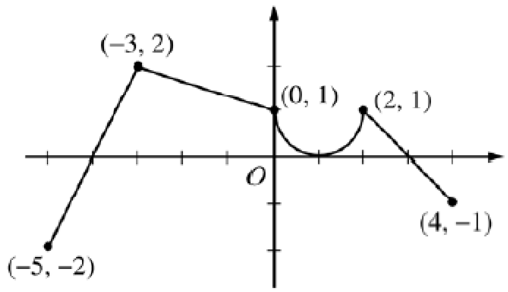
(a) Find $\int_0^{24} v(t)\,dt$. Using correct units, explain the meaning of $\int_0^{24} v(t)\,dt$.

(b) For each of $v'(4)$ and $v'(20)$, find the value or explain why it does not exist. Indicate units of measure.

(c) Let $a(t)$ be the car's acceleration at time t, in meters per second per second. For $0 < t < 24$, write a piecewise-defined function for $a(t)$.

(d) Find the average rate of change of v over the interval $8 \le t \le 20$. Does the Mean Value Theorem guarantee a value of c, for $8 < c < 20$, such that $v'(c)$ is equal to this average rate of change? Why or why not?

2.



Graph of f

The graph of the function f shown above consists of a semicircle and three line segments. Let g be the function given by $g(x) = \int_{-3}^{x} f(t)\,dt$.

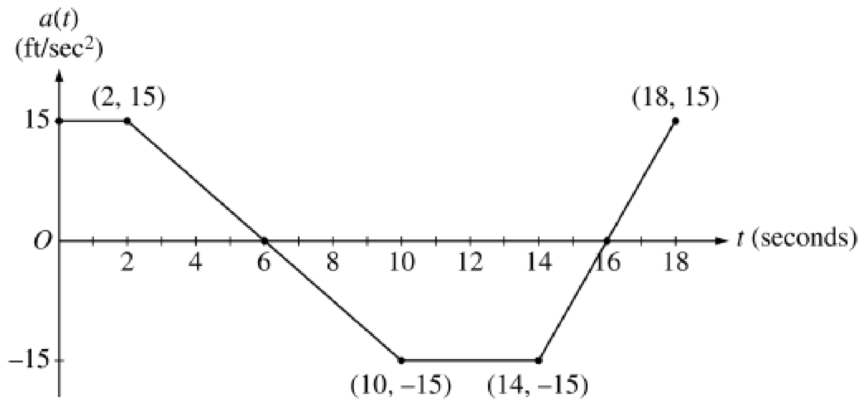
(a) Find $g(0)$ and $g'(0)$.

(b) Find all values of x in the open interval $(-5, 4)$ at which g attains a relative maximum. Justify your answer.

(c) Find the absolute minimum value of g on the closed interval $[-5, 4]$. Justify your answer.

(d) Find all values of x in the open interval $(-5, 4)$ at which the graph of g has a point of inflection.

3.



A car is traveling on a straight road with velocity 55 ft/sec at time $t = 0$. For $0 \le t \le 18$ seconds, the car's acceleration $a(t)$, in ft/sec^2, is the piecewise linear function defined by the graph above.

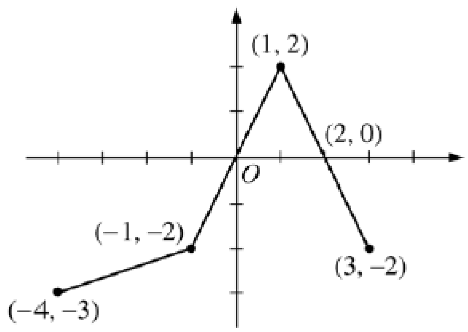
(a) Is the velocity of the car increasing at $t = 2$ seconds? Why or why not?

(b) At what time in the interval $0 \le t \le 18$, other than $t = 0$, is the velocity of the car 55 ft/sec? Why?

(c) On the time interval $0 \le t \le 18$, what is the car's absolute maximum velocity, in ft/sec, and at what time does it occur? Justify your answer.

(d) At what times in the interval $0 \le t \le 18$, if any, is the car's velocity equal to zero? Justify your answer.

4.



Graph of f

The graph of the function f above consists of three line segments.

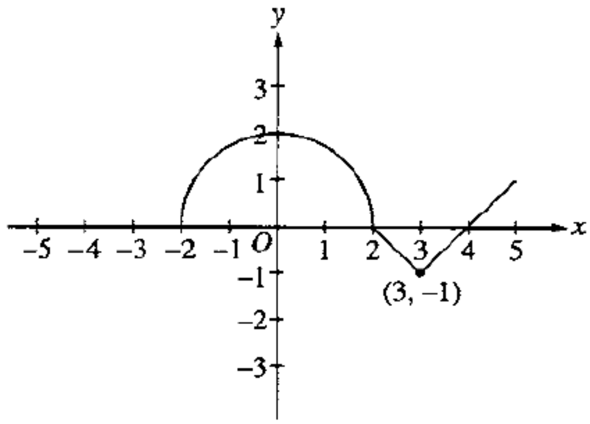
(a) Let g be the function given by $g(x) = \int_{-4}^{x} f(t)\,dt$. For each of $g(-1)$, $g'(-1)$, and $g''(-1)$, find the value or state that it does not exist.

(b) For the function g defined in part (a), find the x-coordinate of each point of inflection of the graph of g on the open interval $-4 < x < 3$. Explain your reasoning.

(c) Let h be the function given by $h(x) = \int_{x}^{3} f(t)\,dt$. Find all values of x in the closed interval $-4 \le x \le 3$ for which $h(x) = 0$.

(d) For the function h defined in part (c), find all intervals on which h is decreasing. Explain your reasoning.

5.



The graph of the function f consists of a semicircle and two line segments as shown above. Let g be the function given by $g(x) = \int_0^x f(t)dt$.

(a) Find $g(3)$.

(b) Find all the values of x on the open interval $(-2,5)$ at which g has a relative maximum. Justify your answer.

(c) Write an equation for the line tangent to the graph of g at $x = 3$.

(d) Find the x-coordinate of each point of inflection of the graph of g on the open interval $(-2,5)$. Justify your answer.

Conceptual Exercises on the Fundamental Theorem of Calculus – Notes

---

Problem 1:

$\int_a^b f(x)dx$ gives the net area between the curve and the x-axis on the interval $[a, b]$. It represents the change in value of the antiderivative of the integrand on that interval.

The derivative of a function represents the slope of the function's curve at that point. A function is differentiable at a point $x = a$ if and only if the function is continuous at $x = a$ and $\lim_{x\to a^-} f'(x) = \lim_{x\to a^+} f'(x)$.

Acceleration is the derivative (rate of change or slope) of velocity.

Be careful to distinguish between average rate of change and the average value of a function. The average rate of change of a function is the slope of the secant line connecting the endpoints of an interval. The average value of a function is the definite integral over an interval, divided by the width of the interval.

The Mean Value Theorem states that if a function is continuous on $[a, b]$ and differentiable on (a, b), then there exists at least one value c, where $a < c < b$, such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Problem 2:

FTC Part II says, $g'(x) = \frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$. What is $g''(x)$ then?

Don't forget that if $b > a$, $\int_b^a f(x)dx = -\int_a^b f(x)dx$

$g(x)$ attains a relative maximum when $g'(x) > 0$

Extreme Value Theorem says a continuous function on a closed interval always has an absolute max and min. These occur at critical values (where $g'(x) = 0$ or DNE) or at endpoints.

$g(x)$ has a point of inflection when $g''(x)$ changes sign.

Problem 3:

Remember that $a(t) = v'(t) = x''(t)$.

Velocity <u>increases</u> when $v'(t) > 0$.

Since $\int_0^t a(x)dx = v(t) - v(0)$ by FTC, $v(t) = v(0) + \int_0^t a(x)dx$, by adding $v(0)$ to both sides. This makes sense because $v(0)$ is the initial velocity and the definite integral gives the change in value of the antiderivative of the integrand (in this case velocity), so the sum of the two gives the velocity for any time t.

A continuous function on a closed interval always has an absolute min and max. Absolute max and min values occur at critical values or endpoints.

Problem 4:

Remember that if $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$, and $g''(x) = f'(x)$

$g(x)$ has a point of inflection when $g''(x)$ changes sign

Remember that $\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$ and $\int_b^a f(x)dx = -\int_a^b f(x)dx$, therefore

$\int_x^a f(x)dx = -\int_a^x f(x)dx$, and $\frac{d}{dx}\left[\int_x^a f(t)dt\right] = -f(x)$.

$h(x)$ is decreasing when $h'(x) < 0$

Problem 5:

Remember that if $g(x) = \int_0^x f(t)dt$, then $g(3) = \int_0^3 f(x)dx$.

$g(x)$ has a relative maximum when $g'(x)$ changes from positive to negative

An equation of the tangent line to $g(x)$ at $(-2,5)$ is

$y - y_0 = m(x - x_0)$ or $y - g(-2) = g'(-2)(x - 5)$

$g(x)$ has a point of inflection when $g''(x)$ changes sign